BENDING OF A SEMI-INFINITE PLATE RESTING ON A LINEARLY DEFORMABLE FOUNDATION

(IZGIB POLUBESKONECHNOI PLITY, LEZHASHCHEI NA LINEINO DEFORMIRUEMOM OSNOVANII)

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A method of deriving the solution of the bending problem for an infinite plate on a linearly deformable foundation of general type is presented in publications of Korenev [1]. The solution of the bending problem for a semi-infinite plate on elastic foundations of some special characteristics was obtained, on the basis of the theory of integral equations of the Wiener-Hopf type [5], in [2,3], and furthermore in [4].

A general method of deriving the solution of the bending problem for a semi-infinite plate on a linearly deformable foundation of general type, with additional loading taken into account, is given below. A detailed presentation is worked out for the case of a semi-infinite beam on the elastic half-plane. A well-known approximate solution of this problem has been given by Gorbunov-Posadov [6]; he himself concedes, however, that the approximate method just mentioned is less accurate than his method of analysing beams of finite length. The reader will find below the results of computations which show the deviations of Gorbunov-Posadov's solution from those obtained in the present publication.

1. Assume a thin semi-infinite plate $(0 \leqslant x < \infty, -\infty < y < \infty)$ of rigidity D to rest on a linearly deformable foundation for which

$$w_0(r) = \int_0^\infty f_0(t) J_0(rt)\, dt \tag{1.1}$$

In this formula $w_0(r)$ denotes the settling of a point on the surface of the foundation at the distance $r = \sqrt{(x^2 + y^2)}$ from the origin of the coordinates, where a unit force is applied. The function $f_0(t)$ can be arbitrary; only its behavior at infinity is supposed to be known.

$$f_0(t) = c_1 t^{-\mu}[1 + 0(1)], \qquad \mu > -\frac{1}{2}, \qquad c_1 = \text{const} \tag{1.2}$$

Equations (1.1) and (1.2) are valid for almost all suggested models of linearly deformable foundations. We note that

$$f_0(t) = (1 - \nu_0^2)(\pi E_0)^{-1} = {}^1/_2\,\theta$$

for the case of a homogeneous half-space.

Furthermore, we assume that the semi-infinite plate is acted upon by the loading $q^+(x, y)$, and the free surface of the foundation by the additional loading $q^-(x, y)$, where

$$q^+(x, y) \equiv 0, \quad (x < 0), \qquad q^-(x, y) \equiv 0, \quad (x \geqslant 0) \tag{1.3}$$

In this case the problem of determining the contact stress $p(x, y)$ and the deflections $w(x, y)$ of the plate is equivalent to that of solving the system

$$\iint\limits_{-\infty}^{\infty} w_0\left(\sqrt{(x-\xi)^2 + (y-\eta)^2}\right)[p(\xi, \eta) + q^-(\xi, \eta)]\,d\xi\,d\eta = w(x, y)$$

$$(-\infty < x,\ y < \infty)$$

$$D\nabla^2\nabla^2 w(x, y) = q^+(x, y) - p(x, y) \qquad (0 \leqslant x < \infty,\ -\infty < y < \infty)$$

followed by fulfilling the boundary conditions for the free edge of the plate

$$\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{x=0} = 0, \qquad \left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]_{x=0} = 0$$

where ν is Poisson's ratio for the plate material.

Passing from the functions $w(x, y)$, $p(x, y)$, $q^+(x, y)$ to their Fourier transforms [7] $w_\lambda(x)$, $p_\lambda(x)$, $q_\lambda^+(x)$, we can reduce the above system to

$$\begin{aligned} &\theta\pi\int\limits_{-\infty}^{\infty} k(|x-\xi|)[p_\lambda(\xi) + q_\lambda^-(\xi)]\,d\xi = w_\lambda(x) \qquad (-\infty < x < \infty) \\ &D\left(\frac{d^2}{dx^2} - \lambda^2\right)^2 w_\lambda(x) = q_\lambda^+(x) - p_\lambda(x) \qquad (0 \leqslant x < \infty) \end{aligned} \tag{1.4}$$

The corresponding form of the boundary conditions for the free edge is then

$$w_\lambda^{(2)}(+0) - \lambda^2\nu w_\lambda(+0) = 0, \qquad w_\lambda^{(3)}(+0) - (2-\nu)\lambda^2 w_\lambda^{(1)}(+0) = 0 \tag{1.5}$$

We use here the notation

$$k(\alpha) = \frac{1}{\pi\theta} \int_{-\infty}^{\infty} w_0(\sqrt{\alpha^2 + \tau^2})\, e^{-i\lambda\tau} dr \tag{1.6}$$

The Fourier transform required for this function is

$$K(u) = \frac{1}{\pi\theta} \iint_{-\infty}^{\infty} w_0(\sqrt{\alpha^2 + \tau^2}) \exp(i\lambda\tau + i\alpha u)\, d\alpha\, d\tau$$

With the aid of procedures given on p. 80 of [7], the expression for $K(u)$ can be transformed into

$$K(u) = \frac{2}{\theta} \int_0^{\infty} r w_0(r) J_0(r\sqrt{u^2 + \lambda^2})\, dr$$

On the basis of (1.1) and with the aid of the formula used for the Hankel transforms [7], we find

$$K(u) = \frac{2f(\sqrt{u^2 + \lambda^2})}{\theta \sqrt{u^2 + \lambda^2}} \tag{1.7}$$

Assuming $\lambda > 0$, we represent the general solution, equal to zero at $x \to \infty$, of the differential equation of the system (1.4) in the form

$$w_\lambda(x) = (a_0 + a_1\lambda x)\, e^{-\lambda x} + \frac{1}{D} \int_{-\infty}^{\infty} g(|x - s|)\, [q_\lambda^+(s) - p_\lambda(s)]\, ds \quad (x \geqslant 0) \tag{1.8}$$

It can be shown that in this presentation

$$g(t) = \frac{1}{4\lambda^3}(1 + \lambda t)\, e^{-\lambda t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm iut} du}{(u^2 + \lambda^2)^2} \tag{1.9}$$

The symbols a_0 and a_1 denote arbitrary real constants.

Equating (1.7) to the first equation of the system (1.4), we arrive at the integral equation

$$\int_0^{\infty} l(|x - s|)\, p_\lambda(s)\, ds = f(x) \qquad (x \geqslant 0) \tag{1.10}$$

with the kernel

$$l(t) = k(t) + c^3 g(t), \qquad c^3 = (\pi\theta D)^{-1} \tag{1.11}$$

and with a right-hand member which can be transformed, with the aid of the inversion theorem [7] and by virtue of (1.3), into

$$f(x) = (a_0 + a_1\lambda x)\,e^{-\lambda x} + \frac{c^3}{2\pi}\int_{-\infty}^{\infty}\left[\frac{Q^+(-\zeta)}{(\zeta^2+\lambda^2)^2} - \frac{2Q^-(-\zeta)\,f\,\sqrt{\zeta^2+\lambda^2})}{c^3\theta\sqrt{\zeta^2+\lambda^2}}\right]e^{i\zeta x}\,d\zeta$$

The functions $Q^{\pm}(u)$ denote here the Fourier transforms of the functions $q_\lambda^{\pm}(x)$.

To solve the obtained integral equation of the Wiener-Hopf type of the first kind, we use the procedure which we have employed in [8,9]. First we solve the equation

$$\int_0^\infty l(|x-s|)\,\chi_\zeta(s)\,ds = e^{i\zeta x} \qquad (x \geqslant 0,\ \mathrm{Im}\,\zeta \geqslant 0) \tag{1.12}$$

If, in the case under consideration, we can find $\chi_\zeta(x)$, then the solution of Equation (1.10) can be obtained by the formula

$$p_\lambda(x) = a_0\,[\chi_\zeta(x)]_{\zeta=i\lambda} - i\lambda a_1\left[\frac{\partial\chi_\zeta(x)}{\partial\zeta}\right]_{\zeta=i\lambda} +$$
$$+\frac{c^3}{2\pi}\int_{-\infty}^{\infty}\left[\frac{Q^+(-\zeta)}{(\zeta^2+\lambda^2)^2} - \frac{2Q^-(-\zeta)\,f\,(\sqrt{\zeta^2+\lambda^2})}{c^3\theta\sqrt{\zeta^2+\lambda^2}}\right]\chi_\zeta(x)\,d\zeta \tag{1.13}$$

The solution of Equation (1.12) is to be derived from the formula

$$\chi_\zeta(x) = \frac{i}{2\pi}\int_\gamma \frac{\psi_\lambda(\zeta)\,\psi_\lambda(u)}{u+\zeta}\,e^{-iux}\,du \tag{1.14}$$

established and proved for integral equations of analogous type, but of the second kind [5]. The function $\psi_\lambda(w)$ must be regular and differ from zero in the upper half of the plane (excluding the point ∞), and satisfy the equation

$$\left[L(u) = \int_{-\infty}^{\infty} l(x)\,e^{ixu}\,dx\right]^{-1} = \psi_\lambda(u)\,\psi_\lambda(-u) \qquad (-\infty < u < \infty) \tag{1.15}$$

Furthermore, its behavior at infinity must be restricted by the condition

$$\psi_\lambda(w) = 0\,(w^\mu) \qquad (\mu < 1,\ w \to \infty) \tag{1.16}$$

By the contour γ is meant the straight line $(-\infty, \infty)$ parallel to the real axis of sufficiently small distance from the latter, more accurately stated, at such a distance that all singular points of the function $\psi_\lambda(w)$ be situated below this straight line. It can be shown by immediate substitution of (1.14) into Equation (1.10), with (1.15), (1.16) taken into account, that (1.10) is actually satisfied [8,9].

In this way the solution of Equation (1.12) is reduced to the problem of finding the function $\psi_\lambda(w)$, or, using the terminology of [5], to the problem of factorizing the function $L(u)$, which in the case under consideration is, according to (1.11), (1.9) and (1.7), of the form

$$L(u)=\frac{2f(\sqrt{u^2+\lambda^2})}{\theta\sqrt{u^2+\lambda^2}}+\frac{c^3}{(u^2+\lambda^2)^2} \tag{1.17}$$

It is possible, as shown in the paper just mentioned, to factorize any function $H(u)$, continuous in the interval $(-\infty, \infty)$, differing from zero and equal to unity when $u\to\pm\infty$. If, moreover, the function $H(u)$ is even, its factorization $H^{-1}(u)=\kappa(u)\kappa(-u)$ is unique and ascertained by the function

$$\varkappa(w)=\exp\left[-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln H(u)}{u-w}\,du\right] \tag{1.18}$$

With this in view, we reduce the function (1.17), which is to be factorized, first to the following form:

$$L(u)=\frac{2c_1}{\theta}(u^2+\lambda^2)^{-\frac{1+\mu}{2}}\left\{\frac{f\sqrt{u^2+\lambda^2}}{c_1(u^2+\lambda^2)^{-\mu/2}}+\frac{(2c_1)^{-1}\theta c^3}{(u^2+\lambda^2)^{(3-\mu)/2}}\right\}$$

Taking (1.2) into account, we now see that the expression within braces satisfies the conditions necessary for its factorization, in accordance with Formula (1.18). The factor before the braces is factorized elementarily. In accordance with the above statements we shall have

$$\psi_\lambda(w)=\left(\frac{2c_1}{\theta}\right)^{-1/2}(\lambda-iw)^{\frac{1+\mu}{2}}X_\lambda(w) \tag{1.19}$$

$$X_\lambda(w)=\exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\ln\left[\frac{f(\sqrt{u^2+\lambda^2})}{c_1(u^2+\lambda^2)^{-\mu/2}}+\frac{(2c_1)^{-1}\theta c^3}{(u^2+\lambda^2)^{3/2-\mu/2}}\right]\frac{du}{u-w}\right\} \tag{1.20}$$

It is shown in [2] that in the case of an elastic homogeneous half-space $[f_0(t)=\theta/2=c_1,\ \mu=0]$ the formulas just obtained become

$$\psi_\lambda(w)=\sqrt{\lambda-iw}\,X_\lambda(w),\qquad X_\lambda(w)=\prod_{j=1}^{3}\chi_j(w;\lambda),\quad \operatorname{Re}\sqrt{\lambda-iw}>0 \tag{1.21}$$

$$\chi_j(w;\lambda)=\chi_j(i\lambda\cos\tau,\lambda)$$

$$=\sqrt{\frac{\cos\tau+1}{\cos\tau+\cos\sigma_j}}\exp\left[\frac{1}{2\pi}\int_{\tau-\sigma_j}^{\tau+\sigma_j}\frac{u}{\sin u}\,du\right],\qquad \sin\sigma_j=a_j/\lambda\qquad (j=1,2,3) \tag{1.22}$$

$$a_1 = c\varepsilon, \quad a_2 = c\varepsilon, \quad a_3 = -c, \quad \varepsilon = e^{i\pi/3} \tag{1.23}$$

2. Expression (1.19) obtained for $\psi_\lambda(w)$ is to be substituted into (1.14), which gives $\chi_\zeta(x)$, and this latter function is to be substituted into (1.13). As a result of the further obvious transformations we obtain the formula

$$p_\lambda(x) = \frac{c^3}{2\pi}\int_\gamma \left[\frac{iP_\lambda(w)}{(w+i\lambda)^2} - I(-w)\right]\psi_\lambda(w)\, e^{-iwx}\, dw \tag{2.1}$$

where

$$I(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{Q^+(-\zeta)}{(\zeta^2+\lambda^2)^2} - \frac{2Q^-(-\zeta)\, f(\sqrt{\zeta^2+\lambda^2})}{c^3\theta\sqrt{\zeta^2+\lambda^2}}\right]\frac{\psi_\lambda(u)}{u-z}\, du \tag{2.2}$$

$$P_\lambda(w) = A_0 + (w+i\lambda)A_1 \tag{2.3}$$

$$c_3 A_0 = a_1 i\lambda\psi_\lambda(i\lambda), \qquad c^3 A_1 = a_0\psi_\lambda(i\lambda) - a_1 i\lambda\psi_\lambda'(i\lambda) \tag{2.4}$$

Here, and everywhere in the following, a prime denotes a derivative.

Substitution of Expression (2.1) for $p_\lambda(x)$ into Formula (1.8) leads, after simple computations, to

$$Dw_\lambda(x) = D(a_0 + a_1\lambda x)e^{-\lambda x} + \frac{c^3}{2\pi i}\int_\gamma \frac{P_\lambda(w)\,\psi_\lambda(w)\, e^{-iwx}}{(w+i\lambda)^2(w^2+\lambda^2)^2}\, dw + $$
$$+ \frac{c^3}{2\pi}\int_\gamma \frac{Q^+(-w) + I(-w)\,\psi_\lambda(w)}{(w^2+\lambda^2)^2}\, e^{-iwx}\, dw \qquad (x \geqslant 0) \tag{2.5}$$

Another expression can be obtained for $w_\lambda(x)$, if (2.1) is substituted not into (1.8) but into the first equation of the system (1.4). The formula obtained in this manner proves to be useful in some cases.

Taking (2.3), (2.4) into account and using the methods of the theory of residues, we can find from (2.5) an expression for $w_\lambda(x)$ and its derivatives $w_\lambda^{(n)}$ for $x \to +0$; computations lead to

$$Dw_\lambda^{(n)}(+0) = a_0\left[c^3\frac{(2n-3)\,\psi_\lambda^2(i\lambda) + 2i\lambda\psi_\lambda'(i\lambda)\,\psi_\lambda(i\lambda)}{16\lambda^{4-n}} + (-\lambda)^n D\right] +$$
$$+ a_1\left[-n(-\lambda)^n D + c^3\frac{(n-2)\,\psi_\lambda^2(i\lambda) - (2n-3)\, i\lambda\psi_\lambda(i\lambda)\,\psi_\lambda'(i\lambda) + 2\lambda^2[\psi_\lambda'(i\lambda)]^2}{16\lambda^{4-n}}\right] +$$
$$+ \frac{c^3}{2\pi}\int_\gamma \frac{Q^+(-w) + I(-w)\,\varphi_\lambda(w)}{(w^2+\lambda^2)^2}(-iw)^n\, dw, \qquad (n = 0, 1, 2, 3) \tag{2.6}$$

We substitute now the expressions for $w_\lambda(x)$ and its derivatives at $x = +0$ into the transformed boundary conditions (1.5). As a result we obtain two equations from which we find the arbitrary constants a_0 and a_1, and this completes the determination of the functions $w_\lambda(x)$ and $p_\lambda(x)$.

The derivation of the final formulas is, however, not possible by means of immediate reduction of the Fourier transform of the functions $w(x, y)$ and $p(x, y)$ in accordance with the formula [7]

$$w(x, y), p(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [w_\lambda(x), p_\lambda(x)]\, e^{-i\lambda y}\, d\lambda$$

since $w_\lambda(x)$ and $p_\lambda(x)$ have been obtained on the assumption that $\lambda > 0$. This condition will be complied with only in the case when the functions $w(x, y)$ and $p(x, y)$ will be simultaneously even or odd with respect to y. Consequently, in the general case we will have to proceed as follows: to resolve the given loadings $q^+(x, y)$ and $q^-(x, y)$ into components symmetrical and skew-symmetrical with respect to the x-axis, i.e.

$$q^+(x, y) = q_1^+(x, y) + q_2^+(x, y), \qquad q^-(x, y) = q_1^-(x, y) + q_2^+(x, y)$$

to find $w(x, y)$ and $p(x, y)$ for even components q_1^+, q_1^- and for odd ones q_2^+, q_2^-, and then to add the results.

It so happens that the functions $p_\lambda(x)$ and $w_\lambda(x)$ which we have derived here are of interest by themselves as well. The reason is that their limiting expressions at $\lambda \to 0$ will represent the solution of the corresponding plane problem, i.e. of the problem of a beam-type plate in bending on a linearly deformable foundation. This follows from the condition that, if the plate is acted upon by the loading $q^+(x, y) = q^+(x) \cos \lambda y$, and the free surface of the foundation by the additional loading $q^-(x, y) = q^-(x) \cos \lambda y$, the contact stress and the deflections of the plate will be

$$p(x, y) = p_\lambda(x) \cos \lambda y, \qquad w(x, y) = w_\lambda(x) \cos \lambda y$$

Thus, denoting by $M(x)$, $Q(x)$, $p(x)$ the bending moment, the shear force and the contact stress, respectively, of the beam-type plate, we verify without difficulty the validity of the formulas

$$M(x) = -\lim_{\lambda \to 0} D w_\lambda^2(x), \qquad Q(x) = -\lim_{\lambda \to 0} D w_\lambda^3(x), \qquad p(x) = \lim_{\lambda \to 0} p_\lambda(x) \quad (2.7)$$

Carrying out these passages to the limits here is by no means a simple operation, however, since according to (2.6) the knowledge of the asymptotic representation of the functions $\psi_\lambda(i\lambda)$ and $\psi_\lambda'(i\lambda)$ at $\lambda \to 0$ will be necessary for that operation, which in turn involves a very detailed analysis of the properties of the function (1.19).

In view of this situation the passage to the limit could be achieved only for the case of the elastic homogeneous half-space. It was found to be more convenient to start not from Formula (2.5) but from the expression for $w_\lambda(x)$, which can be obtained, as already indicated, by substitution of (2.1) into the first equation (1.4). We shall give here this formula. In doing so we restrict ourselves to the case that there is no additional loading [$q^-(x, y) \equiv 0$] and that the semi-infinite plate is acted upon at the point with the abscissa $x = b$ by a concentrated force. Then, instead of (2.2), we shall have

$$I(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-ibu}}{(u^2+\lambda^2)^2}\frac{\psi_\lambda(u)}{u-z}\,du = I_b(z;\lambda) \tag{2.8}$$

Substitution of (2.1), with (2.8) taken into account, into the first equation of the system (1.4) gives, as shown in [2], as a result of transformations indicated there, the desired formula

$$Dw_\lambda^{(n)}(x) = -\frac{ic^4}{\pi}\int_\lambda^\infty \frac{P_\lambda(-ics)(cs+\lambda)(-cs)^n e^{-cxs}\,ds}{\sqrt{cs-\lambda}\,[c^6+(c^2s^2-\lambda^2)^3]\,X_\lambda(ics)} + \tag{2.9}$$

$$+\sum_{j=1}^{2}\frac{P_\lambda(-\alpha_j)\,a_j^3\,(i\alpha_j)^n}{\psi_\lambda(\alpha_j)(i\lambda-\alpha_j)^2}\,r_j e^{i\alpha_j x} - \frac{c^4}{\pi}\int_\lambda^\infty \frac{I_b(ics;\lambda)(c^2s^2-\lambda^2)^{3/2}(-cs)^n e^{-cxs}\,ds}{[c^6+(c^2s^2-\lambda^2)^3]\,\psi_\lambda(ics)} +$$

$$+\,i\sum_{j=1}^{2}\frac{I_b(\alpha_j;\lambda)}{\psi_\lambda(\alpha_j)}\,a_j^3(i\alpha_j)^n r_j e^{i\alpha_j x} + \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{(icu)^n e^{i(x-b)cu}\,du}{\sqrt{c^2u^2+\lambda^2}\,[c^3+(c^2u^2+\lambda^2)^{3/2}]} \qquad (n=2,3)$$

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where*

$$r_j = \frac{(-1)^j a_j}{ic\sqrt{3}\,(a_j+c)\,\alpha_j}, \qquad (j=1,2), \qquad \alpha_j = \sqrt{a_j^2-\lambda^2}, \tag{2.10}$$

$$\operatorname{Im}\sqrt{a_j^2-\lambda^2} > 0 \qquad (j=1,2,3)$$

By varying the path of integration (a procedure described in greater detail in [2] and [9]) into a loop which embraces the ray $(-i\lambda, -i\infty)$,

* The third and the fourth terms of the right-hand side of Equation (2.9) for $w_\lambda^{(n)}(x)$ in [2] contain typographical errors. They are given correctly in the present paper.

Formulas (2.1) and (2.8) can be transformed into

$$p_\lambda(x) = \frac{ic^4}{\pi}\int_\lambda^\infty \frac{P_\lambda(-ics)(c^2s^2-\lambda^2)^{5/2}(cs+\lambda)^2}{\psi_\lambda(ics)[c^6+(c^2s^2-\lambda^2)^3]} e^{-cxs}\,ds +$$

$$+ c^3\sum_{j=1}^{2}\frac{P_\lambda(-\alpha_j)\,a_j^4 r_i}{\psi_\lambda(\alpha_j)(i\lambda-\alpha_j)} e^{i\alpha_j x} + \frac{c^4}{\pi}\int_\lambda^\infty \frac{I_b(ics;\lambda)(c^2s^2-\lambda^2)^{7/2}e^{-cxs}}{\psi_\lambda(ics)[c^6+(c^2s^2-\lambda^2)^3]}\,ds + \tag{2.11}$$

$$+ ic^3\sum \frac{I_b(\alpha_j;\lambda)\,a_j^4}{\psi_\lambda(\alpha_j)} r_j e^{i\alpha_j x} + \frac{c^4}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i(x-b)u}\,du}{c^3+(c^2u^2+\lambda^2)^{3/2}}$$

$$I_b(z;\lambda) = -\frac{ic}{\pi}\int_\lambda^\infty \frac{(c^2s^2-\lambda^2)^{3/2}e^{-bcs}\,ds}{\psi_\lambda(ics)[c^6+(c^2s^2-\lambda^2)^3](z+ics)} + \sum_{j=1}^{2}\frac{r_j}{\psi_\lambda(\alpha_j)}\frac{e^{ib\alpha_j}}{(\alpha_j+z)} \tag{2.12}$$

3. To analyse a semi-infinite beam (beam-type plate) resting on the elastic half-plane (of half-space) and subjected to a concentrated force (loading uniformly concentrated along the infinite line $x = b$) at distance $x = b$ from the end, we pass to the limit $\lambda \to 0$ in Formulas (2.9) to (2.12) in accordance with (2.7). Let us start with function $\chi_i(w;\lambda)$, defined by Formula (1.22). Considering that

$$w = i\lambda\cos\tau = i\lambda\sin\left(\frac{\pi}{2}-\tau\right), \qquad \tau = \frac{\pi}{2} - \sin^{-1}\frac{w}{i\lambda}$$

$$\cos\tau = \frac{w}{i\lambda}, \quad \sigma_j = \sin^{-1}\frac{a_j}{\lambda}, \quad \cos\sigma_j = -\frac{i\alpha_j}{\lambda}, \quad -\frac{\pi}{2} < \mathrm{Re}(\sin^{-1} z) < \frac{\pi}{2}$$

we may write

$$\chi_j(ics;\lambda) = \sqrt{\frac{cs+\lambda}{cs-i\alpha_j}}\exp\left[\frac{1}{2\pi}\int_{\tau-\sigma_j}^{\tau+\sigma_j}\frac{u}{\sin u}\,du\right] \tag{3.1}$$

where

$$\tau \pm \sigma_j = \frac{\pi}{2} - \sin^{-1}\frac{cs}{\lambda} \pm \sin^{-1}\frac{a_j}{\lambda} \tag{3.2}$$

Using, furthermore, the formulas (see pp. 113 and 157 of [10])

$$\sin^{-1}\frac{x\pm iy}{\lambda} = \sin^{-1}\frac{2x}{r+s} \pm i\,\mathrm{Arch}\frac{r+s}{2\lambda}$$

$$\mathrm{Arch}\frac{r+s}{2\lambda} = \ln(r+s) - \ln\lambda - O(\lambda^2), \qquad \lambda\to 0$$

$$r = \sqrt{(x+\lambda)^2+y^2}, \qquad s = \sqrt{(x-\lambda)^2+y^2}, \qquad y > 0$$

we find without difficulty

$$\lim_{\lambda \to 0} \text{arc sim} \frac{x \pm iy}{\lambda} = \text{arc sin} \frac{x}{\sqrt{x^2+y^2}} \pm i (\ln 2\sqrt{x^2+y_2} + i \infty \tag{3.3}$$

We agree in the following to consider a negative (positive) real number to be the limit of a complex number with negative (positive) imaginary part. Computations show that reversal of this stipulation does not affect the final formulas. Starting from this consideration and using Formulas (3.2), (3.3) and (1.23), we find that

$$\begin{aligned} &\tau + \sigma_1 \to \pi/6 - i \ln(s/c), \quad &\tau - \sigma_1 \to -\pi/6 - i\infty. \\ &\tau + \sigma_2 \to \pi/6 - i\infty, \quad &\tau - \sigma_2 \to -\pi/6 - i \ln(s/c) \\ &\tau + \sigma_3 \to -\pi/2 + i\infty, \quad &\tau - \sigma_3 \to \pi/2 - i \ln(s/c) \end{aligned} \tag{3.4}$$

By $\tau + \sigma_1 \to \pi/6 - i \ln s/c$, for example, we mean here

$$\lim_{\lambda \to 0} (\tau + \sigma_1) = \frac{\pi}{6} - \mathrm{i} \ln \frac{s}{c} - \frac{\pi}{2} < \operatorname{Im}(\ln z) < \frac{\pi}{2}$$

We now find easily

$$\chi_1(ics) = \lim_{\lambda \to 0} \chi_1(ics; \lambda) = \sqrt{\frac{cs}{cs - ia_1}} \exp[I(s)] \tag{3.5}$$

where

$$I(s) = \frac{1}{2\pi} \int_{\pi/6 - i\infty}^{\pi/6 - i \ln s/c \pi/6} \frac{u}{\sin u} du = \frac{i}{2\pi} \left(\int_{i\pi/6 - \infty}^{0} + \int_{0}^{i\pi/6 - \ln s/c} \right) \frac{u}{\sinh u} du \tag{3.6}$$

Furthermore, we have on the basis of Cauchy's theorem for analytic functions

$$\frac{i}{2\pi} \int_{i\pi/6 - \infty}^{0} \frac{u}{\sinh u} du = -\frac{i}{2\pi} \int_{0}^{-\infty} \frac{u}{\sinh u} du = \frac{i}{2\pi} \int_{0}^{\infty} \frac{u}{\sinh u} du = \frac{i\pi}{8} \tag{3.7}$$

To find the latter integral it was necessary to use a known formula given on p. 169 of [11]; the handbook [12], p. 171, gives this formula with a typographical error. Taking (3.5) to (3.7) and (1.23) into account, we obtain instead of (3.5)

$$\chi_1(ics) = \sqrt{\frac{s}{s - i\varepsilon}} \exp\left[\frac{i\pi}{8} - \frac{i}{2\pi} \int_{0}^{\ln s + i\pi/6} \frac{u}{\sinh u} du\right]$$

Analogously we obtain, with due attention to (2.10)

$$\chi_2(ics) = \sqrt{\frac{s}{s+i\bar{\varepsilon}}}\exp\left[-\frac{i\pi}{8}+\frac{i}{2\pi}\int_0^{\ln s - i\pi/6}\frac{u}{\sinh u}\,du\right]$$

$$\chi_3(ics) = \sqrt{\frac{s}{s-i}}\exp\left[-\frac{i\pi}{8}+\frac{i}{2\pi}\int_0^{\ln s + i\pi/2}\frac{u}{\sinh u}\,du\right]$$

Consequently, according to (1.21)

$$X_0(ics) = \lim_{\lambda\to 0} X_\lambda(ics) = X(s) = \frac{s^{3/2}}{\sqrt{s^2+\sqrt{3}s+1}}\exp\left[-\frac{i\pi}{8}+\sum_{j=1}^{3} h_j(s)\right] \quad (3.8)$$

where

$$h_{1,2}(s) = \mp\frac{i}{2\pi}\int_0^{\ln s \pm i\pi/6}\frac{u}{\sinh u}\,du, \qquad h_3(s) = \frac{i}{2\pi}\int_0^{\ln s + i\pi/2}\frac{u}{\sinh u}\,dn \quad (3.9)$$

In accordance with (1.21) and (2.10) we have

$$\lim_{\lambda\to 0}\psi_\lambda(ics) = \sqrt{cs}\,X_0(ics) = \sqrt{cs}\,X(s)$$

$$\lim_{\lambda\to 0}\psi_\lambda(\alpha_1) = \sqrt{c}e^{-i\pi/12}X_0(c\varepsilon), \qquad \lim_{\lambda\to 0}\psi_\lambda(\alpha_2) = \sqrt{c}e^{i\pi/12}X_0(-c\bar{\varepsilon}) \quad (3.10)$$

Noting that

$$X_0(c\varepsilon) = X_0[ic\exp(-i\pi/6)], \qquad X_0(-c\bar{\varepsilon}) = X_0\left[ic\exp\left(\frac{i\pi}{6}\right)\right]$$

we substitute into Formula (3.8), instead of s, first $e^{-i\pi/6}$ and then $e^{i\pi/6}$, this leads to

$$X_0(c\varepsilon) = \frac{e^{-i\pi/8}}{\sqrt{6}}, \qquad X_0(-c\bar{\varepsilon}) = \frac{e^{i\pi/8}}{\sqrt{2\sqrt{3}}}\exp\left[-\frac{i}{2\pi}\left(\int_0^{i\pi/3} - \int_0^{i2\pi/3}\right)\frac{u}{\sinh u}\,du\right]$$

The expression within the square brackets equals $-1/4 \ln 3$. In order to check this statement it is necessary to combine the integrals, which appear there, into one with integration limits $(i\pi/3,\ i2\pi/3)$ and to substitute $u = i(1/2\,\pi - v)$; this permits us to reduce that integral to a tabulated one with the integration interval $(0;\ \pi/6)$. In this way we arrive at

$$\lim_{\lambda\to 0}\psi_\lambda(\alpha_1) = \sqrt{c/6}\exp\left(-\frac{5\pi i}{24}\right), \qquad \lim_{\lambda\to 0}\psi_\lambda(\alpha_2) = \sqrt{c/6}\exp\left(\frac{5\pi i}{24}\right) \quad (3.11)$$

Paying attention to (2.10) and (1.23), we find

$$\lim_{\lambda\to 0} r_1 = \tfrac{1}{3} c^{-2} e^{i\pi/3}, \qquad \lim_{\lambda\to 0} r_2 = -\tfrac{1}{3} c^{-2} e^{-i\pi/3} \tag{3.12}$$

Assuming that $X(s)$ is a real function (the proof of this will be given below), we derive from (2.12)

$$\lim_{\lambda\to 0} J_b(ics;\lambda) = J_b(ics) = c^{-7/2}\Big\{-\frac{1}{\pi}\int_0^\infty \frac{\tau^3 e^{-bc\tau}\,d\tau}{\sqrt{\tau}\,(1+\tau^6)\,X(\tau)\,(\tau+s)} +$$
$$+\frac{2\sqrt{6}}{3}\frac{\exp(-\sqrt{3}bc/2)}{1+\sqrt{3}s+s^2}\Big[\sin\Big(\frac{17}{24}\pi+\frac{bc}{2}\Big)+s\sin\Big(\frac{13}{24}\pi+\frac{bc}{2}\Big)\Big]\Big\} \tag{3.13}$$

$$\lim_{\lambda\to 0} J_b(\alpha_1;\lambda) = J_b(c\varepsilon) = c^{-7/2}\Big\{-\frac{1}{\pi}\int_0^\infty \frac{\Big(\tau+\frac{\sqrt{3}}{2}+\frac{i}{2}\Big)\tau^3 e^{-bc\tau}\,d\tau}{(\tau^2+\sqrt{3}\,\tau+1)\sqrt{\tau}\,(\tau^6+1)\,X(\tau)} +$$
$$+\frac{\sqrt{6}}{3}\exp\Big(-\frac{\sqrt{3}}{2}bc\Big)\Big[\frac{1}{\sqrt{3}}\cos\Big(\frac{\pi}{24}+\frac{bc}{2}\Big)+\frac{1}{2}\cos\Big(\frac{5\pi}{24}+\frac{bc}{2}\Big)+$$
$$+i\Big(\frac{1}{2}\sin\Big(\frac{5\pi}{24}+\frac{bc}{2}\Big)-\frac{1}{\sqrt{3}}\sin\Big(\frac{\pi}{24}+\frac{bc}{2}\Big)\Big)\Big]\Big\} = \overline{J}_b(-c\overline{\varepsilon}) \tag{3.14}$$

In order to obtain for the quantities $M(s)$, $Q(x)$, $p(x)$ expressions more convenient for calculation, we introduce a dimensionless abscissa and new arbitrary constants to replace those defined by Formulas (2.4):

$$\xi = cx, \qquad \beta = bc, \qquad B_0 = ic^{3/2}A_0, \qquad B_1 = c^{1/2}A_1 \tag{3.15}$$

Further, we introduce reduced quantities $M^*(\xi)$, $Q^*(\xi)$, $p^*(\xi)$ related to the actual ones by means of the formulas

$$M^*(\xi) = cM(\xi/c), \qquad Q^*(\xi) = Q(\xi/c), \qquad p^*(\xi) = \frac{1}{c}p(\xi/c) \tag{3.16}$$

Turning to Formulas (2.9) and (2.11) and passing to the limit $\lambda \to 0$, with (3.10) to (3.14) and (2.7), (3.15) and (3.16) taken into consideration, we find

$$M^*(\xi) = B_0\Big[\frac{1}{\pi}J^{(0)}(\xi)-\varphi(\xi)\Big]+B_1\Big[\frac{1}{\pi}J^{(1)}(\xi)+\varphi'(\xi)\Big]-\frac{1}{\pi^2}J_2(\xi)-$$
$$-\frac{1}{2\pi}[\varphi'(\xi)J_*^{(0)}(\beta)-\varphi''(\xi)J_*^{(1)}(\beta)-\varphi''(\beta)J_*^{(2)}(\xi)+\varphi'''(\beta)J_*^{(3)}(\xi)]-$$
$$-\frac{2}{3}\Big[\frac{1}{\sqrt{3}}\sin\Big(\frac{\pi}{6}+\frac{\xi-\beta}{2}\Big)+\frac{1}{2}\cos\Big(\frac{\pi}{12}-\frac{\xi+\beta}{2}\Big)\Big]\times$$
$$\times\exp\Big[-\frac{\sqrt{3}}{2}(\xi+\beta)\Big]+M_\infty(\xi-\beta) \tag{3.17}$$

$$Q^*(\xi) = \frac{dM}{d\xi}, \qquad p^*(\xi) = \frac{d^2M^*}{d\xi^2} \tag{3.18}$$

where

$$\varphi(x)=\frac{2\sqrt{6}}{3}e^{-\sqrt{3}x/2}\cos\left(\frac{\pi}{24}+\frac{x}{2}\right) \tag{3.19}$$

$$J^{(n)}(x)=\int_0^\infty \vartheta_n(s,x)\,ds \qquad (n=0,1,2,3)$$

$$J_*^{(n)}(x)=\int_0^\infty \frac{\vartheta_n(s,x)\,ds}{(1+\sqrt{3}S+S^2)} \qquad (n=0,1,2,3,4,5) \tag{3.20}$$

$$J_n x=\int_0^\infty I_\beta(s)\,\vartheta_n(s,x)\,ds \quad (n=2,3,4), \qquad I_\beta(x)=\int_0^\infty \frac{\vartheta_0(s,\beta)\,ds}{(s+x)}$$

$$\vartheta_n(s,x)=\frac{s^{3+n}e^{-sx}}{\sqrt{s}(1+s^6)X(s)}$$

while

$$M_\infty(x)=\frac{1}{\pi}\int_0^\infty \frac{u\cos ux}{1+u^3}\,du \tag{3.21}$$

denotes the reduced bending moment in the infinitely long beam subjected to a concentrated force at $x = 0$.

This function has been tabulated in [6]; in addition, [12] gives its approximation by elementary functions. The same publication gives a procedure for transformation of slowly converging improper integrals of the type (3.21) into very rapidly converging ones. Unfortunately, the present author was not acquainted with Al'perin's work [14], which presents the same method with application to the same integrals. To B.G. Korenev, who called attention to this, the author herewith expresses his gratitude.

The arbitrary constants B_0 and B_1, appearing in (3.17) and (3.18), will be found from the conditions of the free end of the beam, i.e. from $M^*(0) = 0$, $Q^*(0) = 0$. Substituting into the left-hand sides of these equations the values of $M^*(0)$ and $Q^*(0)$ obtained from Formulas (3.17) and (3.18), we arrive at the system

$$\begin{gathered}\left[\frac{1}{\pi}J^{(0)}(0)-\frac{2\sqrt{6}}{3}\cos\frac{\pi}{24}\right]B_0+\left[\frac{1}{\pi}J^{(1)}(0)-\frac{2\sqrt{6}}{3}\cos\frac{\pi}{8}\right]B_1+f_1(\beta)=0\\ -\left[\frac{1}{\pi}J^{(1)}(0)-\frac{2\sqrt{6}}{3}\cos\frac{\pi}{8}\right]B_0-\left[\frac{1}{\pi}J^{(2)}(0)-\frac{2\sqrt{6}}{3}\sin\frac{5\pi}{24}\right]B_1+f_2(\beta)=0\end{gathered} \tag{3.22}$$

The functions $f_{1,2}(\beta)$ will be obtained from the formulas given above for $M^*(\xi)$ and $Q^*(\xi)$, respectively, omitting there the terms containing

B_0 and B_1 and putting $\xi = 0$. We note that the determinant of the system (3.22) equals to unity with accuracy to the fourth decimal.

4. We shall now give a representation of the function $X(s)$ convenient for computation, and at the same time we shall simplify the integrals (3.20). To this end we use the substitution

$$u = \ln t \, (-\tfrac{1}{2}\pi < \operatorname{Im}(\ln z) < \tfrac{1}{2}\pi)$$

in Formulas (3.9)

$$h_{1,2}(\tau) = \mp \frac{i}{\pi} \int\limits_{1}^{\tau \exp(\pm i\pi/6)} \frac{\ln t}{t^2 - 1}\, dt, \quad h_3(\tau) = \frac{i}{\pi} \int\limits_{1}^{\tau \exp(i\pi/2)} \frac{\ln t}{t^2 - 1}\, dt \tag{4.1}$$

Furthermore, we may write, on the basis of Cauchy's theorem (see Fig. 1)

$$\frac{i}{\pi}\Big(\int\limits_{C_2} + \int\limits_{R_\varphi} + \int\limits_{C_1}\Big) \frac{\ln t}{1 - t^2}\, dt = 0 \tag{4.2}$$

This leads, by virtue of (4.1), as easily seen, to

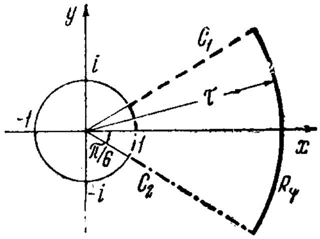

Fig. 1.

$$h_1(\tau) + h_2(\tau) = \frac{i}{\pi} \int\limits_{\tau \exp(-i\pi/6)}^{\tau \exp(i\pi/6)} \frac{\ln t}{1 - t^2} dt$$

$$= -\frac{\tau}{\pi}\Big[i \int\limits_{-\pi/6}^{\pi/6} \frac{\varphi e^{i\varphi}\, d\varphi}{1 - \tau^2 e^{2i\varphi}} + \ln \tau \int\limits_{-\pi/6}^{\pi/6} \frac{e^{i\varphi}\, d\varphi}{1 - \tau^2 e^{2i\varphi}}\Big] \tag{4.3}$$

According to (4.1) we have (Fig. 2)

$$h_3(\tau) = -\frac{i}{\pi}\Big(\int\limits_{C_r''} + \int\limits_{C_2}\Big) \frac{\ln t}{1 - t^2}\, dt \tag{4.4}$$

By corresponding changes in the integration variables we obtain

$$h_3(\tau) = \frac{i}{2\pi} \int\limits_{0}^{i\pi/2} \frac{u}{\sinh u}\, du + \frac{1}{\pi}\int\limits_{1}^{\tau} \frac{\ln s}{1 + s^2}\, ds + \frac{i}{2}\int\limits_{1}^{\tau} \frac{ds}{1 + s^2} \tag{4.5}$$

On the other hand, paying attention to (4.4) and (see Fig. 2) to

$$\frac{i}{\pi}\Big[\int\limits_{C_r'} + \int\limits_{C_1} + \int\limits_{C_R} + \int\limits_{C_2} + \int\limits_{C_r''}\Big] \frac{\ln t}{1 - t^2}\, dt = 0$$

we find

$$h_3(\tau) = \frac{t}{2\pi}\int_0^{-i\pi/2}\frac{u}{\sinh u}\,du - \frac{1}{\pi}\int_1^{\tau}\frac{\ln s}{1+s^2}\,ds + \frac{i}{2}\int_1^{\tau}\frac{ds}{1+s^2} + \frac{\tau}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\ln(\tau e^{i\varphi})\,e^{i\varphi}}{1-\tau^2 e^{2i\varphi}}\,d\varphi \quad (4.6)$$

Taking the sum of (4.5) and (4.6) and carrying out obvious calculations, we obtain

$$h_3(\tau) = \frac{i}{2}\tan^{-1}\tau - \frac{i\pi}{8} + \frac{\tau}{2\pi}\left[i\int_{-\pi/2}^{\pi/2}\frac{\varphi e^{i\varphi}\,d\varphi}{1-\tau^2 e^{2i\varphi}} + \ln\tau\int_{-\pi/2}^{\pi/2}\frac{e^{i\varphi}\,d\varphi}{1-\tau^2 e^{2i\varphi}}\right] \quad (4.7)$$

Finally, using the notations

$$S_\varphi(\tau) = \int_{-\varphi}^{\varphi}\frac{ue^{iu}\,du}{1-\tau^2 e^{2iu}}, \qquad S_\varphi^*(\tau) = \int_{-\varphi}^{\varphi}\frac{e^{iu}\,du}{1-\tau^2 e^{2iu}} \quad (4.8)$$

$$H_0(\tau) = \frac{i\tau}{\pi}\left(\tfrac{1}{2}S_{\pi/2} - S_{\pi/6}\right) + \frac{\tau\ln\tau}{\pi}\left(\tfrac{1}{2}S_{\pi/2}^* - S_{\pi/6}^*\right) \quad (4.9)$$

and taking into account (3.8), (4.3), (4.7), as well as the relation

$$(\tau-i)^{-1/2}\exp\left(\tfrac{1}{2}i\tan^{-1}\tau - \tfrac{1}{4}i\pi\right) = (1+\tau^2)^{-1/4}$$

we find

$$X(\tau) = \tau^{3/2}(\tau^2+\sqrt{3}\tau+1)^{-1/2}(\tau^2+1)^{-1/4}\exp[+H_0(\tau)] \quad (4.10)$$

The function $H(\tau)$ can be expanded into the series

$$H_0(\tau) = \frac{1}{\pi}\sum_{k=0}^{\infty}\left[\frac{(-1)^k - 2\sin[(2k+1)\pi/6]}{2k+1}\ln\tau - \frac{\pi}{3}\,\frac{\cos[(2k+1)\pi/6]}{2k+1} - \frac{(-1)^k - 2\sin[(2k+1)\pi/6]}{(2k+1)^2}\right]\tau^{2k+1} \quad (4.11)$$

convergent for $\tau \leqslant 1$. This series is obtained by means of expansion of the integrand in the integrals (4.8) in terms of ascending powers of τ. Expansion in terms of decreasing powers of τ would give for $H_0(\tau)$ the series

$$H_0(\tau) = \frac{1}{\pi}\sum_{k=0}^{\infty}\left[-\frac{(-1)^k - 2\sin[(2k+1)\pi/6]}{2k+1}\ln\tau - \frac{\pi}{3}\,\frac{\cos[(2k+1)\pi/6]}{2k+1} - \frac{(-1)^k - 2\sin[(2k+1)\pi/6]}{(2k+1)^2}\right]\tau^{-(2k+1)} \quad (\tau \geqslant 1) \quad (4.12)$$

The representation (4.11), (4.12) of the function $H_0(\tau)$ in the entire interval $(0, \infty)$ shows that it is a real function, and the same is true of the function $X(r)$; it leads, furthermore, to the very important relation

$$H_0(\tau)=H_0(\tau^{-1}) \qquad (0\leqslant\tau\leqslant1) \quad (4.13)$$

which permits us to simplify significantly the computation of the integrals (3.20).

Let us illustrate this statement with the example of the integrals $J^{(n)}(\xi)$, $n = 0, 1, 2, 3$. Taking (4.10) into consideration, we subdivide the integration interval into two intervals:

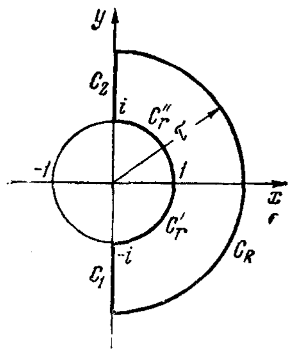

Fig. 2.

$$J^{(n)}(\xi)=\left(\int_0^1+\int_1^\infty\right)e^{-H_0(\tau)}\frac{\sqrt{\tau^2+\sqrt{3}\tau+1}\,\sqrt[4]{\tau^2+1}}{\tau^6+1}\,\tau^{1+n}e^{-\tau\xi}\,d\tau$$

Then we substitute into the second integral $r = s^{-1}$; using (4.13) we find

$$J^{(n)}(\xi)=\int_0^1 F(s)\left[s^{1+n}e^{-s\xi}+\frac{s^{2-n}}{\sqrt{s}}e^{-\xi/s}\right]ds \qquad (n=0,\,1,\,2,\,3) \qquad (4.14)$$

where

$$F(s)=\sqrt{s^2+\sqrt{3}s+1}\,\sqrt[4]{s^2+1}\,(s^6+1)^{-1}e^{-H_0(s)} \qquad (4.15)$$

The remaining integrals (3.20) are to be reduced in the same manner to analogous forms. The function (4.15) will again appear there in the integrands. Table 1 gives its values necessary for the computation of the integrals by means of Simpson's rule, with different degrees of accuracy. The function $H_0(s)$ was computed with the aid of its representation (4.11); for the slowly converging series appearing in the latter it was possible to find the sum

$$\sum_{k=0}^{\infty}\frac{2\sin(2k+1)\pi/6-(-1)^k}{2k+1}\tau^{2k+1}=\tan^{-1}\frac{\tau}{1-\tau^2}-\tan^{-1}\tau \qquad (\tau\leqslant1)$$

$$\frac{1}{3}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi/6}{2k+1}\tau^{2k+1}=\frac{1}{12}\ln\frac{1+\sqrt{3}\tau+\tau^2}{1-\sqrt{3}\tau+\tau^2} \qquad (\tau\leqslant1)$$

The summation of these series was aided by the expansion of $\tan^{-1}$ into a series, as well as by two expansions given on p. 54 of [12].

The simplest way of calculating all integrals (3.20) is to use Simpson's rule, except the integral $J^{(2)}(0)$, for whose computation Simpson's rule cannot be applied in its immediate form. In this case a sufficiently accurate approximation is provided by the function (see Table 1)

$$F(s)=\begin{cases}1+1.320\,s & (0\leqslant s<0.6)\\ -0.8000+7.600\,s-5.500\,s^2 & (0.6\leqslant s\leqslant 1)\end{cases} \tag{4.16}$$

The formulas obtained here permit us to compute the values of M^*, Q^*, p^* for a semi-infinite beam on an elastic half-plane under the action of a concentrated force (Table 2) or a concentrated moment (Table 3) at its end. These tables also give the values of $M^*(\xi)$, $Q^*(\xi)$, $p^*(\xi)$ corresponding to the approximate solution obtained by Gorbunov-Posadov on p. 140 of [6].

TABLE 1.

s	Values of $F(s)$		s	Values of $F(s)$	
	Exact	Acc. to (4.16)		Exact	Acc. to (4.16)
0	1.000	1.000	0.6	1.782	1.780
0.1	1.120	1.132	0.625	1.802	1.802
0.125	1.153	1.165	0.7	1.824	1.825
0.2	1.253	1.264	0.75	1.804	1.806
0.25	1.323	1.330	0.8	1.758	1.760
0.3	1.394	1.396	0.875	1.627	1.639
0.375	1.501	1.495	0.9	1.572	1.585
0.4	1.537	1.528	1	1.297	1.300
0.5	1.674	1.660			

We note that in the case of loading applied at the end of the beam, Formulas (3.17) and (3.18) undergo simplification, because all terms free of B_0 and B_1 must be considered to be zero in this case; for the concentrated moment Formulas (3.16) assume the form

$$M^*(\xi)=M(\xi/c),\qquad Q^*(\xi)=\frac{1}{c}Q(\xi/c),\qquad p^*(\xi)=\frac{1}{c^2}p(\xi/c)$$

In the case of a concentrated force acting at the end of the beam, we have to use in (3.22) as free terms the quantities

$$f_1(\beta)=0,\qquad f_2(\beta)=1$$

while in the case of a concentrated moment we shall have

$$f_1(\beta) = -1, \qquad f_2(\beta) = 0$$

TABLE 2.

ξ	Exact Solution			Solution According to [6]		
	M^*	Q^*	p^*	M^*	Q^*	p^*
0.0	0	−1	∞	0	−1	—
0.2	−0.123	−0.558	1.298	−0.16	−0.61	1.60
0.4	−0.190	−0.246	0.754	−0.25	−0.34	1.16
0.6	−0.227	−0.124	0.452	−0.30	−0.16	0.73
0.8	−0.243	−0.041	0.339	−0.32	−0.05	0.48
1.0	−0.245	0.015	0.227	−0.32	—	0.31
1.2	−0.237	0.052	0.141	−0.31	0.08	0.21
1.4	−0.225	0.076	0.089	−0.29	0.11	0.11
1.6	−0.208	0.089	0.046	−0.26	0.13	0.05
1.8	−0.189	0.095	0.016	−0.34	0.13	0.01
2.0	−0.170	0.096	0.006	−0.21	0.13	−0.02
3.0	−0.086	0.067	−0.038	−0.10	0.09	−0.05
4.0	−0.037	0.033	−0.027	−0.04	0.04	−0.03

TABLE 3.

ξ	Exact Solution			Solution According to [6]		
	M^*	Q^*	p^*	M^*	Q^*	p^*
0.0	1.00	0.00	−∞	1.00	0.00	—
0.2	0.94	−0.43	−0.75	0.97	−0.28	−1.02
0.4	0.84	−0.52	−0.20	0.90	−0.43	−0.51
0.6	0.74	−0.53	0.03	0.81	−0.49	−0.18
0.8	0.63	−0.51	0.15	0.70	−0.51	0.02
1.0	0.53	−0.48	0.21	0.60	−0.49	0.14
1.2	0.44	−0.43	0.25	0.51	−0.46	0.20
1.4	0.36	−0.38	0.25	0.42	−0.41	0.23
1.6	0.29	−0.33	0.24	0.35	−0.36	0.24
1.8	0.23	−0.28	0.23	0.28	−0.32	0.24
2.0	0.18	−0.24	0.21	0.22	−0.27	0.22
3.0	0.03	−0.08	0.11	0.04	−0.10	0.12
4.0	−0.01	−0.01	0.04	−0.01	−0.02	0.05

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